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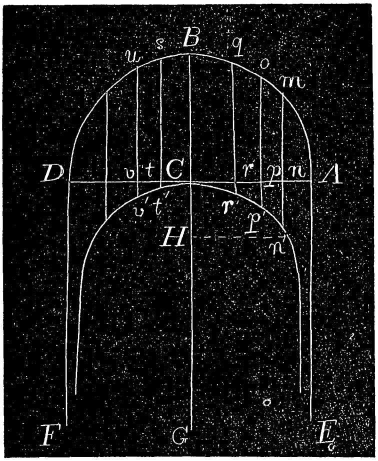
CURVE OF LOGARITHMIC SINES,

BY PROFESSOR L. G. BARBOUR, RICHMOND, KENTUCKY.

LET ABD be a semicircle, C its centre, radius $= 1$; and let fall from any number of points in the circumference sines, as mn, op, qr, BC, st, uv, &c. At A and D draw the tangents AE and DF. Prolong the sines below the axis AD until $nn' = \log mn$, $pp' = \log op$ &c. Through the extremities of these lines at n', p', r', C, t', v' &c., draw a curve. It may be appropriately called the curve of logarithmic sines; $rr' = \log qr$; $pp' = \log op$; $nn' = \log mn$.



Taking C as the origin of rectangular co-ordinates, and designating the lines rr', pp', nn' &c., generally, by y, and the cosines Cr, Cp, Cn, &c., by x, we have

$$y = \log \surd(1 - x^2)$$

as the equation of the curve.

It is obvious that the curve has two symmetrical branches; that it is tangent to AD at C; for $BC = 1$, $\therefore \log BC = 0$; and that the tangent lines AE and DF are asymptotes. The logarithms are negative, as they should be for quantities less than 1. When the sine $= 0$, its logarithm $= -\infty$.

Length of Arc.—To get this we write $y = \log\surd(1 - x^2) = \frac{1}{2}\log(1 - x^2)$;

$$\therefore\ dy = -\frac{xdx}{1 - x^2}. \quad \text{Putting arc} = z,\ dz = dx\sqrt{\left(\frac{1 - x^2 + x^4}{1 + 2x^2 + x^4}\right)}.$$

$$z = \int_0^a dx\sqrt{\left(\frac{1 - x^2 + x^4}{1 + 2x^2 + x^4}\right)} = a + \frac{a^3}{6} + \frac{7a^5}{40} + \frac{17a^7}{112} + \frac{139a^9}{1152} + \frac{263a^{11}}{2816} + \&c.$$

Surface of Revolution.—Let $s =$ surface of revolution on CG as an axis; $\therefore ds = 2\pi x\, dz$.

$$s = \int_0^x \pi\left[xdx + \frac{1}{2}x^3dx + \frac{7}{8}x^5dx + \frac{17}{16}x^7dx + \frac{139}{128}x^9dx + \&c.\right]$$

$$= \pi\left[\frac{x^2}{2} + \frac{x^4}{8} + \frac{7x^6}{48} + \frac{17x^8}{128} + \frac{139x^{10}}{1280} + \frac{263x^{12}}{3072} + \&c.\right]$$

Plane Areas.—We distinguish two of these; the interior, included within the arc $n'p'r'Ct'v'$; and the exterior, lying between the tangents AE, DF, the diameter ACD, and the curve. As both these areas are symmetrically divided by the axis CG, let us calculate first the interior area included between the axis CG and the curve $Cr'p'n'$.

Designating this area by I, $dI = xdy = x\,d\log\surd(1-x^2) = x\tfrac{1}{2}d\log(1-x^2)$

$$= \frac{-x^2dx}{1-x^2} = dx - \frac{dx}{1-x^2} = dx - \frac{1}{2}\cdot\frac{dx}{1-x} - \frac{1}{2}\cdot\frac{dx}{1+x}.$$

$$\therefore \quad I = \int\frac{-x^2dx}{1-x^2} = x + \tfrac{1}{2}\log(1-x) - \tfrac{1}{2}\log(1+x).$$

Therefore the double interior area $= 2I = 2x + \log(1-x) - \log(1+x)$. $x = 0$ gives area $= 0$, $\therefore c = 0$. $x = 1$ gives area $= 2 + \infty - \log 2 = \infty$.

Next, the exterior area, between $Crpn$, nn' and $Cr'p'n'$, designated by E. $dE = ydx = \tfrac{1}{2}\log(1-x^2)dx$. Integrating by parts,

$$\int\tfrac{1}{2}\log(1-x^2)dx = \tfrac{1}{2}\log(1-x^2)x - \int\frac{-x^2dx}{1-x^2}. \quad \therefore E = xy - I.$$

This is evidently correct, for the area of any rectangle as $Cnn'H = xy$, and it is equal to the sum of $E + I$.

When $x = 0$, $E = 0$. When $x = 1$, $E = \log 2 - 1$; $2E$, the double exterior area, $= 2\log 2 - 2$. This is a remarkably simple expression, and furnishes another case, in addition to the "Witch" of Agnesi and a few other curves, of a finite area enclosed by lines some of which at least are infinitely long.

The proof however may be acceptable to some of my readers.

$$E = xy - I = xy - x - \tfrac{1}{2}\log(1-x) + \tfrac{1}{2}\log(1+x)$$

$$2E = x\log(1-x^2) - 2x - \log(1-x) + \log(1+x)$$

$$= \left.\begin{matrix} x\log(1+x) + x\log(1-x) - 2x \\ + \log(1+x) - \log(1-x) \end{matrix}\right\}$$

$$= (1+x)\log(1+x) - (1-x)\log(1-x) - 2x.$$

Since when $x = 1$, $1 - x = 0$, and $\log(1-x) = \infty$, the limit of their product is not apparent at the first glance; but it may be found by developing $\log(1-x)$ and then performing the multiplication by $(1-x)$. Thus:

$$\left.\begin{matrix} \log(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \&c. \\ -x\log(1-x) = x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{3}x^4 + \tfrac{1}{4}x^5 + \&c. \end{matrix}\right\}$$

So long as x is less than 1, the upper of these series is greater than the lower; but when $x = 1$, they are just equal and the term $(1-x)\log(1-x) = 0$. $\therefore$ for $x = 1$, $2E = (1+x)\log(1+x) - 2x = 2\log 2 - 2$.

It will be observed that I have throughout used the Napierian logarithms. In any system of logarithms,

$$2E = 2\log 2 - 2M,$$

the logarithm of 2 being taken in that system. In the Napierian system, $M = 1$, and $\log 2 = .693147$; $\therefore E = .693147 - 1 = -.306853$.

Logarithmoids. — If the curve be revolved about CG as an axis, it will describe a logarithmoid of revolution. $dv = \pi x^2dy$; therefore

$$v = \pi\int x^2\tfrac{1}{2}d\log(1-x^2) = \pi\int\frac{-x^3dx}{1-x^2} = \pi(\tfrac{1}{2}x^2+y)$$

in any system of logarithms. This may be called the interior volume.

An exterior volume is described by the exterior area *Cnn′* revolving about *CG*. Designating this by *E. V.*, $d\,E.\,V. = 2\pi yxdx = \pi\log(1-x^2)xdx$.

Integrating by parts; $\int udv = uv - \int vdu$. Let $u = \log(1-x^2)$, $\therefore$

$$du = \frac{-2xdx}{1-x^2}.$$

Let $xdx = dv$, $\therefore$ $v = \tfrac{1}{2}x^2$

$\therefore \int\pi\log(1-x^2)xdx = \pi x^2y - \tfrac{1}{2}x^2 - y = -\pi[(1-x^2)\tfrac{1}{2}\log(1-x^2)+\tfrac{1}{2}x^2$.

When $x=1$, it can be shown very much as in the former case that $(1-x)\times\tfrac{1}{2}\log(1-x^2) = 0$; $\therefore$ when $x = 1$, $E.\,V. = \tfrac{1}{2}\pi$.

Now the volume of the hemisphere described by revolving the quadrant *ABC* about *BC* as an axis, $= \tfrac{2}{3}\pi$; $\therefore$ $E.\,V. = \tfrac{3}{4}$ hemisphere.

Singular Points.— One has already been noted, the point of tangency of the curve with the axis of abscissas. It is characterized by

$$\frac{dy}{dx} = 0. \quad \text{But } \frac{dy}{dx} = \frac{-x}{1-x^2},$$

which is verified by $x = 0$.

Another is the point where a tangent to the curve makes an angle of 45° with the axes. Here $\frac{dy}{dx} = \frac{-x}{1-x^2} = -1$. The sign of the 1 is negative in the 4th quadrant. $\therefore x^2+x=1$; $\therefore 1 : x :: 1+x : 1$; $\therefore 1-x : x :: x : 1$. That is, the abscissa of the point at which a tangent line, making an angle of 45° with either axis, touches the curve, is the greater segment of the radius when divided in extreme and mean ratio according to Euclid. Hence the abscissa = the chord of a decagon, = twice the natural sine of 18° = .61804.

Radius of Curvature.—The equation is

$$r = -\frac{\sqrt{(1-x^2+x^4)^3}}{1-x^4}.$$

When $x = 0$ $r = -1$, $\therefore$ the centre of the osculatory circle is on the axis of y, at the distance 1 below the centre of the original circle.

When $x = 1$, $r = -\infty$.

The Anthoid. — By revolvig the curve *Cr′p′n′* about the tangent line *AE* as an axis, we get a curved surface of a new pattern. The volume generated by the revolution of the surface *n′ p′ r′ C–A*, may from its resemblance to flowers be calld the Anthoid.—I leave these two last as problems for the readers of the ANALYST.